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Disks in trivial braid diagrams

Patrick Dehornoy*

Laboratoire de Mathématiques Nicholas Oresme, CNRS UMR 6139, Université de Caen, 14032 Caen, France

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Abstract

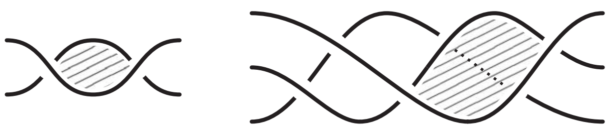
We show that every trivial 3-strand braid diagram contains a disk, defined as a ribbon ending in opposed crossings. Under a convenient algebraic form, the result extends to every Artin–Tits group of dihedral type, but it fails to extend to braids with 4 strands and more. The proof uses a partition of the Cayley graph and a continuity argument.

MSC: 57M25; 20F36

Keywords: Braid diagram; Disk; Isotopy; Cayley graph; Garside monoid

1. Introduction

Let us say that a braid diagram is trivial if it represents the unit braid, i.e., if it is isotopic to an unbraided diagram. Consider the following simple trivial diagrams:



We see that these digrams contain a *disk*, defined as an embedded ribbon ending in crossings with opposite orientations (the striped areas). Below is another trivial braid diagram containing a disk: here the shape is more complicated, but we still have the property that the third strand does

* Corresponding author.

E-mail address: dehornoy@math.unicaen.fr (P. Dehornoy).

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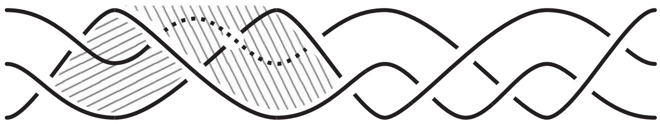
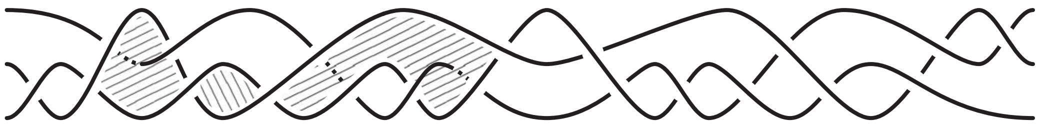
not pierce the disk.



Finally, let us display a more intricate example involving a disk: here the third strand pierces the ribbon, but it does it so as to make a topologically trivial handle through the disk, so, up to an isotopy, we still have an unpierced disk.



A few tries should convince the reader that most trivial braid diagrams seem to contain at least one disk in the sense above—a precise definition will be given below—and make the following question natural:

Question 1.1. *Does every trivial braid diagram* (*with at least one crossing*) *contain a disk*?

Our aim is to answer the question by proving

Proposition 1.2. *The answer to Question* 1.1 *is positive in the case of* 3*-strand braids*, *i.e.*, *every trivial* 3*-strand braid diagram with at least one crossing contains a disk*. *It is negative in the case of* 4 *strands and more*.

As for the negative part, it is sufficient to exhibit a counter-example, what will be done at the end of Section 2 (see Fig. 2).

As for the positive part, the argument consists in going to the Cayley graph of the braid group and using a continuity result, which itself relies on the properties of division in the braid monoid B_n^+. The argument works in every Artin–Tits group of spherical type, and we actually prove the counterpart of (the positive part of) Proposition 1.2 in all Artin–Tits groups of type $I_2(m)$.

One should keep in mind that we are interested in braid diagrams, not in braids: up to an isotopy, all braid diagrams we consider can be unbraided. What makes the question nontrivial is that isotopy may change the possible disks of a braid diagram completely, so that it is hopeless to trace the disks along an isotopy. For instance, the reader can check that applying one type III Reidemeister move in the braid diagram of Fig. 2 suffices to let one disk appear.

2. Disks and removable pairs of letters

Definition (Fig. 1). Assume that D is an n-strand braid diagram, which is the projection of a three-dimensional geometric braid β consisting of n disjoint curves connecting n points $P_1, \ldots, P_n$ in the plane $z = 0$ to n points $P'_1 \ldots, P'_n$ in the plane $z = 1$. For $1 \leqslant i, j < n$, we say that D is an (i, j)-*disk* if D begins with a crossing of the strands starting at P_i and P_{i+1}, it finishes with a

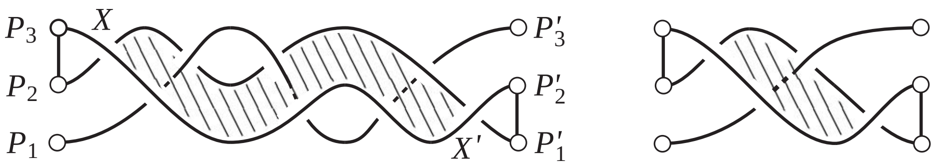


Fig. 1. A (2,1)-disk (left), and a diagram that is *not* a (2,1)-disk (right): the third strand pierces the ribbon made by the first two strands; it is convenient in the formal definition to appeal to the points P_i and P'_j, but, in essence, the disk is the part lying between the crossings denoted X and X'.

crossing of opposite orientation of the strands ending at P'_j and P'_{j+1}, and the figure obtained from β by connecting P_i to P_{i+1} and P'_j to P'_{j+1} is isotopic to the union of $n-2$ curves and the boundary of a disk disjoint from these curves.

This definition is directly reminiscent of the notion of a *life disk* in a singular braid introduced in [15]: another way to state that D is a disk is to say that, when one makes the initial and the final crossings in D singular—with the convention that the first crossing is replaced with a "birth" singular crossing, while the last one, which is supposed to have the opposite orientation, is replaced with a "death" singular crossing—then the resulting figure is a life disk.

We shall address Question 1.1 using the braid group B_n and the geometry of its Cayley graph. As is standard, braid diagrams will be encoded by finite words over the alphabet $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$, using σ_i to encode the elementary diagram where the $(i+1)$th strand crosses over the ith strand. For instance, the first three diagrams above are coded by $\sigma_1\sigma_1^{-1}$, $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1\sigma_2^2\sigma_1\sigma_2^2\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}$, respectively.

We denote by $\equiv$ the equivalence relation on braid words that corresponds to braid isotopy. As is well known, $\equiv$ is the congruence generated by the pairs $(\sigma_i\sigma_j,\ \sigma_j\sigma_i)$ with $|i-j| \geqslant 2$ and $(\sigma_i\sigma_j\sigma_i,\ \sigma_j\sigma_i\sigma_j)$ with $|i-j| = 1$, together with $(\sigma_i\sigma_i^{-1}, \varepsilon)$ and $(\sigma_i^{-1}\sigma_i, \varepsilon)$, where ε denotes the empty word.

Proposition 2.1. *A braid diagram is an (i,j)-disk if and only if it is encoded in a word of the form $\sigma_i^e w \sigma_j^{-e}$ with $e = \pm 1$ and $\sigma_i^e w \sigma_j^{-e} \equiv w$.*

Proof. Assume that D is an (i,j)-disk. By definition, D is encoded in some braid word of the form $\sigma_i^e w \sigma_j^{-e}$ with $e = \pm 1$. Moreover, we can assume that, after an isotopy, the strands of D starting at positions i and $i+1$ make an unpierced ribbon. Then, the initial σ_i^e crossing may be pushed along that ribbon, so as to eventually cancel the final σ_j^{-e} crossing. Hence D is isotopic to the diagram obtained by deleting its first and last crossings, i.e., we have $\sigma_i^e w \sigma_j^{-e} \equiv w$.

Conversely, assume that D is encoded in $\sigma_i^e w \sigma_j^{-e}$ and $\sigma_i^e w \sigma_j^{-e} \equiv w$ holds. Then we have $\sigma_i^e w \equiv w\sigma_j^e$. By Theorem 2.2 of [16], this implies that D contains a ribbon connecting $[i, i+1] \times 0$ to $[j, j+1] \times 1$ that is, up to an isotopy, disjoint from the other strands. Hence, with our current definition, D is an (i,j)-disk. □

Thus we are led to introduce:

Definition. A braid word of the form $\sigma_i^e w \sigma_j^{-e}$ with $e = \pm 1$ is said to be a *removable pair of letters* if $\sigma_i^e w \sigma_j^{-e} \equiv w$ holds.

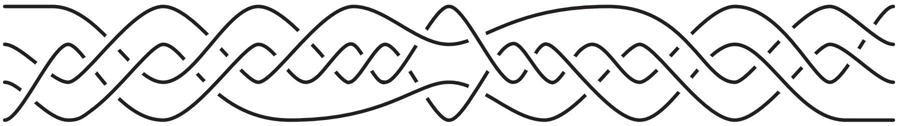


Fig. 2. A trivial 4-strand braid diagram containing no disk.

With this notion, Question 1.1 is equivalent to

Question 2.2. *Does every nonempty trivial braid word contain a removable pair of letters*?

Speaking of "removable pair" is natural here: indeed, saying that a braid word w' contains a removable pair $\sigma_i^e w \sigma_j^{-e}$ implies that w' is equivalent to the word obtained from w' by replacing the subword $\sigma_i^e w \sigma_j^{-e}$ with w, i.e., by deleting the end letters σ_i^e and σ_j^{-e}. Observe that the notion of a removable pair of letters actually makes sense for any group presentation: we shall use it in a more general context in Section 5 below.

As there exist efficient algorithms for deciding braid word equivalence, it is easy to systematically search the possible removable pairs in a braid word, and an experimental approach of Question 2.2 is possible. Random tries would suggest a positive answer, but this is misleading: for instance, the 4 strand braid word

$$\sigma_1^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_2^{-3}\sigma_1^{-2}\sigma_3^{2}\sigma_2^{3}\sigma_1\sigma_2^{2}\sigma_1\sigma_3\sigma_2^{2}\sigma_3\sigma_1\sigma_2^{2}\sigma_3$$

contains no removable pair of letters, and, therefore, the associated braid diagram, which is displayed in Fig. 2, contains no disk. This establishes the negative part of Proposition 1.2.

3. The valuation of a pure simple element

The proof of (the positive part of) Proposition 1.2 relies on partitioning the Cayley graph of B_3 using integer parameters connected with division in the braid monoid B_3^+. The construction is not specific to the braid group B_3, nor is it either specific to braid groups: actually, it is relevant for all spherical type Artin–Tits groups, and, more generally, for all Garside groups in the sense of [11].

A monoid G^+ is said to be a *Garside monoid* if it is cancellative, 1 is the only invertible element, any two elements admit a left and a right least common multiple, and G^+ contains a Garside element, defined as an element whose left and right divisors coincide, they generate the monoid, and they are finite in number. If G^+ is a Garside monoid, it embeds in a group of fractions. A group G is said to be a *Garside group* if G can be expressed in at least one way as the group of fractions of a Garside monoid.

Typical examples of Garside monoids are the braid monoids B_n^+ [17], and, more generally, the Artin–Tits monoids A^+ of spherical type, i.e., those Artin–Tits monoids such that the associated Coxeter group W is finite. In this case, the image of the longest element of W under the canonical section of the projection of A^+ onto W is a Garside element in A^+. In the particular case of B_n^+, one obtains the half-twist braid Δ_n. So, the braid groups B_n, and, more generally, the Artin–Tits groups of spherical type, are Garside groups. Let us mention that a given group may be the group of fractions of several Garside monoids: for instance, the braid groups B_n admit a second Garside structure, associated with the Birman–Ko–Lee monoid of [3]—see [1,18] for similar results involving other Artin–Tits groups. Still another Garside structure for B_3 involves the submonoid generated by σ_1 and $\sigma_1\sigma_2$, a Garside monoid with presentation $\langle a, b; aba = b^2\rangle$, hence not of Artin–Tits type.

Assume that G^+ is a Garside monoid. Then every element x in G^+ admits finitely many expressions as a product of atoms (indecomposable elements), and the supremum $\|x\|$ of the length of these decompositions, called the *norm* of x, satisfies $\|xy\| \geqslant \|x\| + \|y\|$ and $\|x\| \geqslant 1$ for $x \neq 1$. Then there exists in G^+ a unique Garside element of minimal norm; this element is traditionally denoted Δ, and its (left and right) divisors are called the *simple* elements of G^+.

We shall start from two technical results about division in Garside monoids—as shown in [11], these results also happen to be crucial in the construction of an automatic structure [14,6,7,12]. For x, y in a Garside monoid G^+, we denote by $x \setminus y$ the unique element z such that xz is the right lcm of x and y, and we write $y \preccurlyeq z$ (*resp.* $z \succcurlyeq y$) to express that y is a left (*resp.* right) divisor of z.

Lemma 3.1. *Assume that G^+ is a Garside monoid, that y, z are elements of G^+ and that every simple right divisor of yz is a right divisor of z. Let x be an arbitrary element of G^+, and let $y' = x \setminus y$ and $z' = (y \setminus x) \setminus z$. Then every simple right divisor of $y'z'$ is a right divisor of z'.*

Proof. *Let* $x' = y \setminus x$ and $x'' = z \setminus (y \setminus x)$. By definition of a right lcm, we have $xy' = yx'$, and $x'z' = zx''$. Moreover, 1 is the only common right divisor of y' and x'. Assume that s is a simple right divisor of $y'z'$. Then we have $xy'z' \succcurlyeq s$, hence, $yzx'' \succcurlyeq s$. Let $s'x''$ be the left lcm of s and x''. Then $yzx'' \succcurlyeq s$ implies $yzx'' \succcurlyeq s'x''$, hence $yz \succcurlyeq s'$. Moreover, s being simple implies that s' is simple as well, as shows an induction on the minimal number p such that x'' can be decomposed into the product of p simple elements. Then, the hypothesis of the lemma implies $z \succcurlyeq s'$, and, therefore, $zx'' \succcurlyeq s$, i.e., $x'z' \succcurlyeq s$. It follows that s is a right divisor of the right lcm of $y'z'$ and $x'z'$, which is z' since 1 is the only common right divisor of y' and x'. □

Lemma 3.2. *Assume that G^+ is a Garside monoid, that y, z, x are elements of G^+, and that every simple right divisor of yz is a right divisor of z. Then $y \not\preccurlyeq x$ implies $yz \not\preccurlyeq xt$ for every simple element t of G^+.*

Proof. We assume $yz \preccurlyeq xt$, and aim at proving $y \preccurlyeq x$. Let $y' = x \setminus y$, and $z' = (y \setminus x) \setminus z$. By construction, we have $y'z' = x \setminus (yz)$, and $yz \preccurlyeq xt$ implies $y'z' \preccurlyeq t$, so, in particular, $y'z'$ must be simple. By Lemma 3.1, every simple right divisor of $y'z'$ is a right divisor of z', so we deduce $z' \succcurlyeq y'z'$, which is possible for $y' = 1$ only, i.e., for $y \preccurlyeq x$. □

Now, the idea is to consider, for each element of a Garside group G and each simple element s of G^+, the maximal power of s that divides a given element. We begin with the monoid.

Definition. Assume that G^+ is a Garside monoid. We say that a simple element s of G^+ is *pure* if s is the maximal simple right divisor of s^k, for every k. If s is a pure simple element of G^+, we define the (*left*) *valuation* $v_s(x)$ of s in x to be the maximal k satisfying $s^k \preccurlyeq x$.

In the braid monoid B_n^+, each generator σ_i, as well as the Garside element Δ_n—and, more generally, each simple braid which is an lcm of generators σ_i—is a pure simple element. If G^+ is an arbitrary

Garside monoid, the Garside element Δ is always pure by definition, but the atoms or their lcms need not be pure in general: for instance, in the monoid $\langle a, b; aba = b^2\rangle^+$, the atom b is not pure, as b^2 is simple.

Lemma 3.3. *Assume that G^+ is a Garside monoid and that s is a pure simple element of G^+. Then, for every x in G^+, we have*

$$v_s(x) \leqslant v_s(xt) \leqslant v_s(x) + 1, \tag{3.1}$$

whenever t is a simple element of G^+; more specifically, for $t = \Delta$, we have

$$v_s(x\Delta) = v_s(x) + 1. \tag{3.2}$$

Proof. First $s^k \preccurlyeq x$ implies $s^k \preccurlyeq xt$ for every t, hence $v_s(x) \leqslant v_s(xt)$. On the other hand, assume $s^{k+1} \not\preccurlyeq x$. By hypothesis, every right divisor of s^{k+2} is a right divisor of s. Applying Lemma 3.2 with $y = s^{k+1}$ and $z = s$, we deduce $s^{k+2} \not\preccurlyeq xt$, hence $v_s(xt) \leqslant v_s(x) + 1$, and (3.1) follows.

As Δ is simple, (3.1) implies $v_s(x\Delta) \leqslant v_s(x) + 1$. On the other hand, let ϕ be the automorphism of G^+ defined for z a simple element by $\phi(z) = (z \setminus \Delta) \setminus \Delta$ (see [11]). Then $z\Delta = \Delta\phi(z)$ holds for every z. Now assume $s^k \preccurlyeq x$. We find

$$x\Delta = s^k x'\Delta = s^k \Delta\phi(x') = s^{k+1}(s \setminus \Delta)\phi(x'),$$

hence $s^{k+1} \preccurlyeq x\Delta$, and, therefore, $v_s(x\Delta) > v_s(x)$, hence (3.2). □

We now extend the maps v_s from a Garside monoid G^+ to its group of fractions G. As Δ is a common multiple of all atoms in G^+, every element of G can be expressed as $x\Delta^k$ with $x \in G^+$ and $k \in \mathbb{Z}$. Unless we require that k be maximal, the decomposition need not be unique. However, we have the following result:

Lemma 3.4. *Assume that G^+ is a Garside monoid, x, x' are elements of G^+, and we have $x\Delta^k = x'\Delta^{k'}$ in the group of fractions G of G^+. Then, for each pure simple element s of G^+, we have $v_s(x) + k = v_s(x') + k'$.*

Proof. Assume for instance $k \leqslant k'$, say $k' = k + m$. Then we have $x\Delta^k = x'\Delta^m\Delta^k$ in G, hence $x = x'\Delta^m$ in G^+ (we recall that G^+ embeds in G). Using Lemma 3.3 m times, we obtain $v_s(x'\Delta^m) = v_s(x') + m$ for every s, hence $v_s(x) = v_s(x') + m$, i.e., $v_s(x) + k = v_s(x') + k'$. □

Then the following definition is natural:

Definition. Assume that G^+ is a Garside monoid, G is the group of fractions of G^+, and s is a pure simple element of G^+. Then, for x in G, the (*left*) *valuation* $v_s(x)$ of s in x is defined to be $v_s(z) + k$, where $x = z\Delta^k$ is an arbitrary decomposition of x with $z \in G^+$ and $k \in \mathbb{Z}$.

Example 3.5. Let $G = B_3$ and $x = \sigma_1^{-1}\sigma_2$. We can also write $x = \sigma_2\sigma_1^2\Delta_3^{-1}$. We have $v_{\sigma_1}(\sigma_2\sigma_1^2) = 0$ and $v_{\sigma_2}(\sigma_2\sigma_1^2) = 1$, so we find $v_{\sigma_1}(x) = 0 - 1 = -1$, and $v_{\sigma_2}(x) = 1 - 1 = 0$.

It is now easy to see that the inequalities of Lemma 3.3 remain valid in the group.

Proposition 3.6. *Assume that G is the Garside group associated with a Garside monoid G^+, and that s is a pure simple element of G^+. Then, for every element x in G, and every simple element t in G^+, we have*

$$v_s(x) \leqslant v_s(xt) \leqslant v_s(x) + 1, \tag{3.3}$$

for $t = \Delta$, we have

$$v_s(x\Delta) = v_s(x) + 1.$$

Proof. Assume $x = y\Delta^k$ with $y \in G^+$. We have $xt = y\Delta^k t = y\phi^{-k}(t)\Delta^k$. Then $y\phi^{-k}(t)$ belongs to G^+, hence we have $v_s(x) = v_s(y) + k$ and $v_s(xt) = v_s(y\phi^{-k}(t)) + k$. As $\phi^{-k}(t)$ is a simple element of G^+, Lemma 3.3 gives

$$v_s(y) \leqslant v_s(y\phi^{-k}(t)) \leqslant v_s(y) + 1,$$

so (3.3) follows. The result for $t = \Delta$ is obvious, since we obtain $x\Delta = y\Delta^{k+1}$, hence $v_s(x\Delta) = v_s(y) + k + 1 = v_s(x) + 1$ directly. □

Inequality (3.3) is the algebraic socle on which we shall build in the sequel.

4. Partitions of the Cayley graph

From now on, we restrict to Artin–Tits groups, i.e., we consider presentations of the form

$$\langle S; \mathrm{prod}(\sigma, \tau, m_{\sigma,\tau}) = \mathrm{prod}(\tau, \sigma, m_{\sigma,\tau}) \text{ for } \sigma \neq \tau \text{ in } S \rangle, \tag{4.1}$$

where $\mathrm{prod}(\sigma, \tau, m)$ denotes the alternated product $\sigma\tau\sigma\tau\ldots$ with m factors, and $m_{\sigma,\tau} \geqslant 2$ holds. Moreover, we restrict to the spherical type, i.e., we assume that the Coxeter group obtained by adding to (4.1) the relation $\sigma^2 = 1$ for each σ in S is finite. Then the monoid A^+ defined by (4.1) is a Garside monoid, and the group A defined by (4.1) is the group of fractions of A^+.

In this case, each generator σ in S is pure, since σ^2 is not simple and σ is the right gcd of σ^2 and Δ. Hence, each element x of the group A has a well-defined valuation $v_\sigma(x)$ for each σ in S, and we can associate to x the valuation sequence $(v_\sigma(x); \sigma \in S)$.

Example 4.1. Consider the case of B_3. There are two atoms, namely σ_1 and σ_2. The valuation sequence associated with σ_1 is $(1, 0)$, while the one associated with $\sigma_1^{-1}\sigma_2$ is $(1, -1)$, as was seen above. Observe that the influence of right multiplication on the valuation sequence may be anything that is compatible with the constraints of (3.3). For instance, σ_1, $\sigma_1\sigma_2$, and $\sigma_1\sigma_2^2\sigma_1$ all admit the valuation sequence $(1, 0)$, while the valuation sequences of $\sigma_1 \cdot \sigma_2$, $\sigma_1 \cdot \sigma_1$, $\sigma_1\sigma_2 \cdot \sigma_1$, and $\sigma_1\sigma_2^2\sigma_1 \cdot \sigma_2$ are $(1, 0)$, $(2, 0)$, $(1, 1)$, and $(2, 1)$, respectively.

Using the valuation sequence, we can partition the group A, hence, equivalently, its Cayley graph, into disjoint regions according to the values of the valuations. For our current purpose, we shall consider a coarser partition, namely the one obtained by taking into account not the values of the valuations, but their relative positions only. Let us say that two n-tuples of integers $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ are *order-equivalent* if $k_i = k_j$ and $k'_i = k'_j$ (*resp.* $k_i < k_j$ and $k'_i < k'_j$) hold for the same

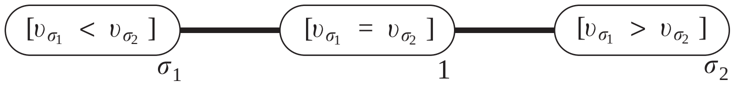


Fig. 3. The 3 types of braids in B_3.

pairs (i, j). The equivalence class of a tuple $(k_1, \ldots, k_n)$ will be called its *order-type*. For instance, there are 3 order-types of pairs, corresponding to pairs (k_1, k_2) with $k_1 < k_2$, $k_1 = k_2$, and $k_1 > k_2$, respectively. Similarly, there are 13 order-types of triples, and, in the general case of n-tuples, the number of order-types is the nth ordered Bell number $\sum_{p=1}^{n} a_p p^n$ with $a_p = \sum_{q=0}^{n-p} (-1)^q \binom{p+q}{q}$.

Definition. Assume that A is an Artin–Tits group of spherical type with presentation (4.1). For x in A, the *type* of x is defined to be the order-type of the sequence $(v_\sigma(x); \sigma \in S)$.

So, there are 3 types of braids in B_3, according to whether the value of v_{σ_1} is smaller than, equal to, or bigger than the value of v_{σ_2}. These types will be denoted $[v_{\sigma_1} < v_{\sigma_2}]$, $[v_{\sigma_1} = v_{\sigma_2}]$ and $[v_{\sigma_1} > v_{\sigma_2}]$. Thus, saying that a braid β in B_3 is of type $[v_{\sigma_1} > v_{\sigma_2}]$ means that there are "more σ_1's than σ_2's at the left of β". For instance, the type of σ_1 is $[v_{\sigma_1} > v_{\sigma_2}]$, while that of σ_2 and of σ_1^{-1} is $[v_{\sigma_1} < v_{\sigma_2}]$ and that of 1 or Δ_3^k is $[v_{\sigma_1} = v_{\sigma_2}]$.

Proposition 3.6 immediately leads to constraints on how the type may change under right multiplication by a simple element.

Proposition 4.2. *Assume that A is an Artin–Tits group of spherical type. Say that two types T, T' are neighbours if there exist $(k_1, \ldots, k_n)$ in T and $(k'_1, \ldots, k'_n)$ in T' such that $k'_i - k_i$ is either* 0 *or* 1 *for every i, or is either* 0 *or* -1 *for every i. Then, for every x in A and every simple element t of A^+, the type of $xt^{\pm 1}$ is a neighbour of the type of x.*

We display in Figs. 3 and 4 the graph of the neighbour relation for order-types of pairs and of triples—as well as examples of 3- and 4-strand braids of the corresponding types. We see in Fig. 3 that the types $[v_{\sigma_1} > v_{\sigma_2}]$ and $[v_{\sigma_1} < v_{\sigma_2}]$ are not neighbours, since, starting with a pair (k_1, k_2) with $k_1 > k_2$ and adding 1 to k_1 or k_2, we can obtain (k'_1, k'_2) with $k'_1 \geqslant k'_2$, but not with $k'_1 < k'_2$. As a consequence, we cannot obtain a braid of type $[v_{\sigma_1} < v_{\sigma_2}]$ by multiplying a braid of type $[v_{\sigma_1} > v_{\sigma_2}]$ by a single simple braid or its inverse: crossing the intermediate type $[v_{\sigma_1} = v_{\sigma_2}]$ is necessary. Similarly, we can see on Fig. 4 that, for instance, going from type $[v_{\sigma_1} > v_{\sigma_2} = v_{\sigma_3}]$ to type $[v_{\sigma_1} < v_{\sigma_2} = v_{\sigma_3}]$ necessitates that one goes through at least one of the intermediate types $[v_{\sigma_1} = v_{\sigma_2} < v_{\sigma_3}]$, $[v_{\sigma_1} = v_{\sigma_2} = v_{\sigma_3}]$ or $[v_{\sigma_1} = v_{\sigma_3} < v_{\sigma_2}]$.

5. Loops in the Cayley graph

We are now ready to establish that every nonempty trivial 3-strand braid word contains at least one removable pair of letters. The geometric idea of the proof is as follows: a trivial word corresponds to a loop in the Cayley graph of B_3, and we can choose the origin of that loop so that it contains vertices of types $[v_{\sigma_1} < v_{\sigma_2}]$ and $[v_{\sigma_1} > v_{\sigma_2}]$. But then Proposition 4.2 tells us that one cannot jump

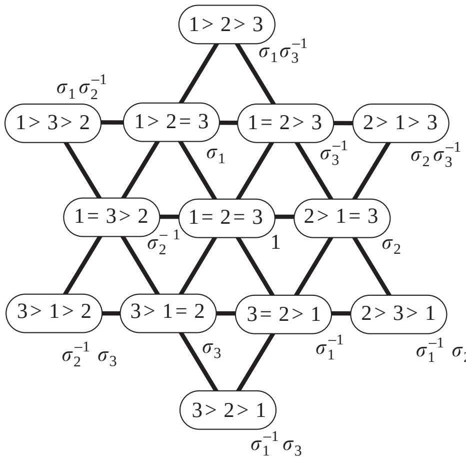


Fig. 4. The 13 types of braids in B_4—here i stands for v_{σ_i}.

from the region $[v_{\sigma_1} < v_{\sigma_2}]$ to the region $[v_{\sigma_1} > v_{\sigma_2}]$ without crossing the separating region, i.e., $[v_{\sigma_1} = v_{\sigma_2}]$. This means that some subword of w must represent a power of Δ_3, and it is easy to deduce a removable pair of letters.

Actually, we shall prove a more general statement valid for every Artin–Tits group with two generators, i.e., for every Artin–Tits group of type $I_2(m)$—the case of B_3 corresponding to $m = 3$.

Proposition 5.1. *Assume that A is an Artin–Tits group of type $I_2(m)$, i.e., A admits the presentation $\langle \sigma_1, \sigma_2; \sigma_1\sigma_2\sigma_1\sigma_2 \cdots = \sigma_2\sigma_1\sigma_2\sigma_1 \ldots \rangle$, where both sides of the equality have length m. Then every nonempty word on the letters $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}$ representing 1 in A contains a removable pair of letters.*

We begin with two auxiliary results.

Lemma 5.2. *Assume that G is a group generated by a set S, that w is a trivial word on $S \cup S^{-1}$ (i.e., w represents 1 in G), and some cyclic conjugate of w contains a removable pair of letters. Then w contains a removable pair of letters.*

Proof. Assume that we have $w = uv$ and $\sigma^e w' \tau^{-e}$ is a removable pair of letters in vu, with $\sigma, \tau \in S$, and $e = \pm 1$. Let us write $vu = w_1 \sigma^e w' \tau^{-e} w_2$. If $w_1 \sigma^e w' \tau^{-e}$ is a prefix of v, or if $\sigma^e w' \tau^{-e} w_2$ is a suffix of u, then $\sigma^e w' \tau^{-e}$ is a subword of w, and the result is obvious. Otherwise, we have $w' = v'u'$ with $\sigma^e v'$ a suffix of v and $u' \tau^{-e}$ a prefix of u, hence $v = w_1 \sigma^e v'$ and $u = u' \tau^{-e} w_2$. By construction, $\tau^{-e} w_2 w_1 \sigma^e$ is a subword of uv, i.e., of w. Let us use $\equiv$ for the congruence that defines G. By hypothesis, we have $uv \equiv \varepsilon$ and $\sigma^e v' u' \tau^{-e} \equiv v'u'$, hence $\tau^e u'^{-1} v'^{-1} \sigma^{-e} \equiv u'^{-1} v'^{-1}$. We deduce

$$w_2 w_1 \equiv \tau^e u'^{-1} u v v'^{-1} \sigma^{-e} \equiv \tau^e u'^{-1} v'^{-1} \sigma^{-e} \equiv u'^{-1} v'^{-1} \equiv u'^{-1} u v v'^{-1} \equiv \tau^{-e} w_2 w_1 \sigma^e,$$

which shows that $\tau^{-e} w_2 w_1 \sigma^e$ is a removable pair of letters in w. $\square$

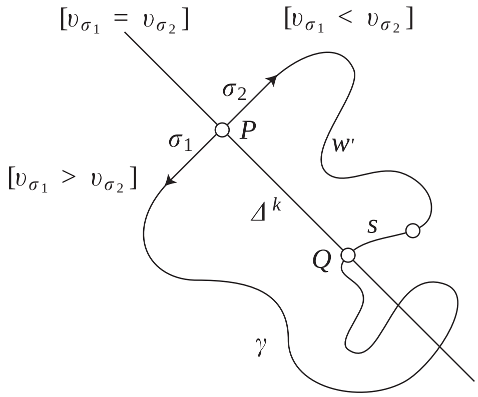


Fig. 5. Proof of Proposition 5.1: a loop must intersect the diagonal at least twice.

Lemma 5.3. *Let A be an Artin–Tits group with presentation* (4.1). *Assume that σ, τ belong to S, s belongs to $S \cup S^{-1}$, and w is a word on $S \cup S^{-1}$ such that $\tau w s \equiv \mathrm{prod}(\sigma, \tau, m_{\sigma,\tau})^k$ holds and τw represents an element of the region $[v_\sigma < v_\tau]$. Then either $\sigma^{-1}\tau w s$ or $\tau w s$ is a removable pair of letters.*

Proof. For u a word on $S \cup S^{-1}$, let $\bar{u}$ denote the element of A represented by u. Let us write m for $m_{\sigma,\tau}$. By hypothesis, we have $v_\sigma(\overline{\tau w s}) = v_\tau(\overline{\tau w s}) = k$. Assume first that mk is even. Then there are two possibilities for s only, namely, $s = \sigma$, and $s = \tau^{-1}$. Indeed, $\overline{\tau w s}$ is $\mathrm{prod}(\sigma, \tau, m)^k$, so $s = \rho^{\pm 1}$ with $\rho \neq \sigma, \tau$ would imply

$$v_\sigma(\overline{\tau w}) = v_\sigma(\overline{\tau w s}) = v_\tau(\overline{\tau w s}) = v_\tau(\overline{\tau w}),$$

while $s = \sigma^{-1}$ and $s = \tau$ would imply

$$v_\sigma(\overline{\tau w}) \geqslant v_\sigma(\overline{\tau w s}) = v_\tau(\overline{\tau w s}) \geqslant v_\tau(\overline{\tau w}),$$

all contradicting the hypothesis $v_\sigma(\overline{\tau w}) < v_\tau(\overline{\tau w})$.

Now, for $s = \sigma$, we find

$$\sigma^{-1}\tau w \sigma \equiv \sigma^{-1}\mathrm{prod}(\sigma, \tau, m)^k \equiv \mathrm{prod}(\sigma, \tau, m)^k\, \sigma^{-1} \equiv \tau w \sigma \sigma^{-1} \equiv \tau w,$$

i.e., $\sigma^{-1}\tau w s$ is a removable pair. Similarly, for $s = \tau^{-1}$, we find

$$\tau w \tau^{-1} \equiv \mathrm{prod}(\sigma, \tau, m)^k \equiv \tau^{-1}\mathrm{prod}(\sigma, \tau, m)^k \tau \equiv \tau^{-1}\tau w \tau^{-1}\tau \equiv w,$$

i.e., $\tau w s$ is a removable pair. The argument is similar when mk is odd, the possible values of s now being σ^{-1} and τ instead of σ and τ^{-1}. $\square$

Proof of Proposition 5.1 (Fig. 5). Assume that w is a nonempty word on the letters $\sigma_1^{\pm 1}$, $\sigma_2^{\pm 1}$ representing 1. Necessarily w contains the same number of letters with exponent $+1$ and with exponent -1, so it must contain a subword of the form $s^{-1}t$ or st^{-1} with $s, t \in \{\sigma_1, \sigma_2\}$. Assume for instance that w contains a subword of the form $s^{-1}t$; the argument in the case of st^{-1} would be similar. The case $s = t$ is trivial (then $s^{-1}s$ is a removable pair of letters of w, and we are done), so, up to a symmetry, we can assume that $s^{-1}t$ is $\sigma_1^{-1}\sigma_2$.

The word w specifies a path γ in the Cayley graph of G, and, by hypothesis, γ is a loop. Let P be the point of γ corresponding to the middle vertex in the subword $\sigma_1^{-1}\sigma_2$ considered above. By Lemma 5.2, we can assume that P is the origin of γ without loss of generality.

Now, let us follow γ starting from P: as the first letter is σ_2, the path γ enters the region $[v_{\sigma_1} < v_{\sigma_2}]$. At the other end, the last letter of γ is σ_1^{-1}, which means that, before ending at P, the path γ comes from the region $[v_{\sigma_1} > v_{\sigma_2}]$. So γ goes from the region $[v_{\sigma_1} < v_{\sigma_2}]$ to the region $[v_{\sigma_1} > v_{\sigma_2}]$. By Proposition 4.2, γ must cross the separating region $[v_{\sigma_1} = v_{\sigma_2}]$ at least once. This means that there must exist at least one second point Q in γ with type $[v_{\sigma_1} = v_{\sigma_2}]$. Now—and this is where we use the hypothesis that A is of Coxeter type $I_2(m)$—the only elements of A of this type are the powers of the element Δ, i.e., of $\mathrm{prod}(\sigma_1, \sigma_2, m)$. So we deduce that (a cyclic conjugate of) w must contain a subword $\sigma_1^{-1}\sigma_2 w' s$ such that $\sigma_2 w' s$ is equivalent to a power of $\mathrm{prod}(\sigma_1, \sigma_2, m)$ and $\sigma_2 w'$ represents an element of the region $[v_{\sigma_1} < v_{\sigma_2}]$. Then Lemma 5.3 implies that either $\sigma_1^{-1}\sigma_2 w' s$ or $\sigma_2 w' s$ is a removable pair of letters. □

Remark 5.4. It is known [2,5,8] that the Cayley graph of any Garside group is traced on some flag complex of the form $\mathscr{X} \times \mathbb{R}$, where the $\mathbb{R}$-component corresponds to powers of Δ. In the case of an Artin–Tits group of type $I_2(m)$, the space $\mathscr{X}$ is an m-valent tree. A loop γ in the Cayley graph projects onto a loop in the tree, so the projection necessarily goes twice through the same vertex, which means that γ contains vertices that are separated by a power of Δ, and we can deduce the existence of a removable pair of letters as above.

6. Special cases

As the counter-example of Fig. 2 shows, a trivial 4-strand braid word need not contain any removable pair of letters. However, partial positive results exist, in particular when we consider words of the form $u^{-1}v$, with u, v positive words representing a divisor of Δ.

The following result is an easy consequence of the classical Exchange Lemma for Coxeter groups ([4], Lemma IV.1.4.3) rephrased for Artin–Tits monoids.

Lemma 6.1. *Assume that A^+ is an Artin–Tits monoid of spherical type, σ, τ are atoms of A^+, and we have $\sigma \not\preccurlyeq x$ and $\sigma \preccurlyeq x\tau \preccurlyeq \Delta$. Then we have $x\tau = \sigma x$.*

Indeed, let π denote the bijection of the divisors of Δ in A^+ to the corresponding Coxeter group W and ℓ denote the length in W. Then $\sigma \preccurlyeq x\tau$ implies $\ell(\pi(\sigma x\tau)) < \ell(\pi(x\tau))$. Hence the minimal decomposition of $\pi(\sigma x\tau)$ is obtained from that of $\pi(x\tau)$ by removing one generator, which cannot come from x for, otherwise, we would obtain $\ell(\pi(\sigma \not\preccurlyeq x)) < \ell(\pi(x))$ by cancelling τ and contradict σx. So we must have $\pi(\sigma x\tau) = \pi(x)$, hence $\pi(\sigma x) = \pi(x\tau)$, in W, and $\sigma x = x\tau$ in A^+.

Proposition 6.2. *Assume that A^+ is an Artin–Tits monoid of spherical type, and w is a nonempty trivial word of the form $u^{-1}v$ with u, v positive simple words. Then w contains at least one removable pair of letters.*

Proof. For w a positive word, let $\overline{w}$ denote the element of A^+ represented by w. Now, let σ be the first letter in u. By hypothesis, we have $\sigma \preccurlyeq \bar{v}$. Let $v'\tau$ be the shortest prefix of v such that $\sigma \preccurlyeq \overline{v'\tau}$

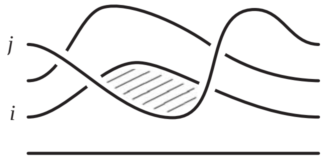


Fig. 6. Disk in a trivial braid diagram coded by $u^{-1}v$ with u, v simple.

is true. Then, by definition, we have $\sigma \not\preccurlyeq \overline{v'}$, and, as $\bar{v}$ is supposed to be simple, so is $\overline{v'\tau}$. We can therefore apply Lemma 6.1, and we obtain $\sigma v' \equiv v'\tau$, hence $\sigma^{-1}v'\tau \equiv v'$. Thus $\sigma^{-1}v'\tau$ is a removable pair of letters in w. □

Remark 6.3. In the case of braids, a direct geometric argument also gives Proposition 6.2. Indeed, if u and v are positive braid words representing simple braids, then the braid diagrams coded by u and v can be realised as the projections of three-dimensional figures where the ith strand entirely lives in the plane $y = i$: the simplicity hypothesis guarantees that no altitude contradiction can occur, as any two strands cross at most once [10]. So the same is true for the braid coded by $u^{-1}v$, provided we require that the strand living in the plane $y = i$ is the one at position i after u^{-1}. Now, let i be the least index such that the ith strand is not a straight line, and let j be the least index such that the jth strand crosses the ith strand. Then, necessarily, the ith and the jth strands make a disk, as they must return to their initial position if $u^{-1}v$ represents 1 (Fig. 6).

Proposition 6.2 does not extend to arbitrary trivial negative–positive words, i.e., of the form $u^{-1}v$ with u, v positive: the hypothesis that u and v represent a simple braid is essential.

An easy method for producing equivalent positive braid words is as follows: starting with a seed consisting of two positive words u, v, we can complete them into equivalent words—i.e., we can find a common right multiple for $\bar{u}$ and $\bar{v}$—by using the word reversing technique of [9], which gives two positive words u', v' so that both uv' and vu' represent the right lcm of $\bar{u}$ and $\bar{v}$. Then, by construction, $v'^{-1}u^{-1}vu'$ represents 1. By systematically enumerating all possible seeds (u, v), we obtain a large number of negative–positive trivial braid words in which possible removable pair of letters can be investigated.

One obtains in this way very few counter-examples, i.e., trivial braid words with no removable pair of letters. In the case of B_4, there exists no counter-example with seeds of length at most 4, and there exists only one counter-example among the 29,403 pairs of length 5 words, namely the one of Fig. 2, which is associated with the seed $(\sigma_1^2\sigma_2^3, \sigma_3^2\sigma_2^3)$. The situation is similar with longer seeds, and for B_n with $n \geqslant 5$. This explains why random tries have little chance to lead to counter-examples, and raises the question of understanding why there seems to almost always exist disks in trivial braid diagrams.

Finally, let us mention a connection with the (open) question of unbraiding every trivial braid diagram in such a way that all intermediate diagrams have at most as many crossings as the initial diagram—as is well known, there is no solution in the case of knots when the number of crossings is considered, but there is now a solution when the complexity is defined in a more subtle way [13]. Assume that a method for detecting removable pairs of letters has been choosen. Then one obtains an unbraiding algorithm by starting with an arbitrary braid word and iteratively removing removable pairs of letters until no one is left. If the answer to Question 1.1 were positive, this

Fig. 7. A ribbon in the counter-example of Fig. 2.

algorithm would always succeed, in the sense that it would end with the empty word if and only if the initial word is trivial. Note that the number of iteration steps is always bounded by half the length of the initial word. In the case of 4 strands and more, the answer to Question 1.1 is negative, so the above algorithm is *not* correct. In addition, it must be kept in mind that, in any case, the algorithm requires a subroutine detecting removable pairs: we can appeal to any solution of the braid word problem, but, then, the algorithm gives no new solution to that word problem, nor does it either answer the question of length-decreasing unbraiding as long as there is no length-preserving method for proving an equivalence of the form $\sigma_i^e w \sigma_j^{-e} \equiv w$.

As trivial diagrams without disk seem to be rare, it might happen that, in some sense to be made precise, the above method almost always works. It can be observed on Fig. 7 that the braid diagram of Fig. 2, which contains no disk, contains an actual ribbon, in the sense that no isotopy is needed to let this ribbon appear. By merging the two strands bordering this ribbon, one obtains a 3-strand diagram—namely, the last example in Section 1—which contains a disk. Improving the unbraiding method so as to include such a strand merging procedure might make it work for still more cases.

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